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Top trading cycles☆

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1. Introduction

The history of top trading cycle (TTC) algorithms begins with the first issue of the *Journal of Mathematical Economics*, which contained the striking paper of Shapley and Scarf (1974), "On Cores and Indivisibility".[1] TTC has subsequently been explored in a growing literature, starting with Roth and Postlewaite (1977).

Shapley and Scarf were concerned with the core and competitive equilibria of a game in which indivisible goods were to be traded, without the aid of any divisible good that could serve as money. Specifically, they considered a model with n traders, each of whom owns one unit of some unique good. Traders have purely ordinal preferences over the goods (which Shapley and Scarf sometimes called "houses"), and no trader has use for more than one good. Let $\omega = \{\omega_1, \ldots, \omega_n\}$ be the initial endowment of goods, where ω_i denotes the good initially owned by the ith trader. Denote the ith trader's strict preferences over the goods by P_i and her weak preference over the goods by R_i. An allocation is any permutation of the initial endowment ω, whose ith component is the good that trader i possesses at the close of the market.[2] Let S be a subset of N. Then an allocation x **(strongly) dominates** an allocation y if there is some coalition S such that *(i)* $\{x_i | i \in S\} = \{\omega_i | i \in S\}$ and *(ii)* $x_i \, P_i \, y_i$ for all $i \in S$. That is, x dominates y if there is some coalition S that, by trading their initial endowments among themselves, can secure for each member of the coalition an item they prefer to the one they receive at y. The core of the game is defined to be the set of undominated outcomes. Shapley and Scarf provided a nonconstructive proof that the core of this game must be non-empty, by showing that the game was "balanced" in the sense required by an earlier theorem of Scarf.

And then, stunningly, and explicitly as an afterthought, they also provided a constructive proof of the nonemptiness of the core by introducing Top Trading Cycles (TTC), saying:

"After the proof in [section] 4 had been discovered, David Gale pointed out to the authors a simple constructive method for finding competitive prices in this market, and hence a point in its core. The following is based on his idea.

Let $R \subseteq N$, and define a ***top trading cycle for*** *R to be any set S, $\emptyset \subset S \subseteq R$, whose s members can be indexed in a cyclic order:*

$$S = \{i_1, i_2, \ldots, i_s = i_0\},$$

in such a way that each trader i_v likes the i_{v+1}'st good at least as well as any other good in R. It is evident that every nonempty $R \subseteq N$ has at least one top trading cycle, for we may start with any trader in R and construct a chain of best-liked goods that eventually must come back to some earlier element.[3]

Using this idea, we can partition N into a sequence of one or more disjoint sets:

$$N = S^1 \cup S^2 \cup \cdots \cup S^p,$$

by taking S^1 to be any top trading cycle for N, then taking S^2 to be any top trading cycle for $N - S^1$, then taking S^3 to be any top trading cycle for $N - (S^1 \cup S^2)$, and so on until N has been exhausted. We can now construct a payoff vector x by carrying out the indicated trades within each cycle. That is, if $i = i_v^j \in S^j$, then x_i is i's utility for the good of trader i_{v+1}^j. We assert (1) that x, so constructed, is in the core, and (2) that a set of competitive prices exists for x.

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[1] See also the companion article in this issue (Afacan et al., 2024): "Housing markets since Shapley and Scarf", by Afacan, Hu, and Li. Readers who were born near the turn of the 21st Century should be forewarned that 50th anniversaries come unexpectedly soon.

[2] Agents care only about their own goods, and not about the whole allocation of goods. For a model in which agents' preferences are over entire allocations rather than over their own allocation, see Hong and Park (2022).

[3] A top trading cycle may consist of a single trader!

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To establish (1), let S be any coalition. Consider the first j such that $S \cap S^j \neq \phi$. Then we have

$$S \subseteq S^j \cup S^{j+1} \cup \cdots \cup S^p = N - (S^1 \cup \cdots \cup S^{j-1}).$$

Let $i \in S \cap S^j$. Then i is already getting in x the highest possible payoff available to him in S. No improvement is possible for him, unless he deals outside of S. Hence S cannot strictly improve, and it follows that x is in the core".[4]

The simplicity of TTC and of this proof can be made clear (and taught e.g. to high school students) by recasting it as an entirely verbal proof (without symbols or subscripts). Consider the directed graph whose nodes are the people and houses. The edges of the graph go from each house to its original owner ("each house points to the person who originally owned it"), and from each person to his most preferred house ("each person points to his favorite house"). A cycle can be found by starting with an arbitrary person and following the outbound edges until the first time some person is encountered twice (the cycle is the closed path that begins and ends with this person). This cycle is removed from the set of nodes (and each person in the cycle is allocated the house to which they pointed), and the process repeats, one cycle at a time with the remaining houses and people, until all houses have been allocated, which must occur because at least one house is allocated at each step. (At each step, the graph must have at least one cycle, which may consist of a person pointing to his own house.) To see that the resulting allocation is in the core, note first that no member of the first cycle that was removed is available to participate in a dominating coalition because they all received their most preferred house, and so cannot improve upon what they already have been allocated. But no dominating coalition can include people from the second cycle without the participation of people in the first cycle, because only houses that were allocated in the first cycle are among those that people removed in the second cycle might prefer (since they received their most preferred house among those remaining after the prior cycle was removed). In the same way, this argument applies, one at a time, to each subsequent cycle, none of whose members can be part of a coalition that can improve all members' assignments by trading their initial endowments among themselves.

The paper by Roth and Postlewaite (1977) was (as far as we know) the first followup to Shapley and Scarf (1974), and it derived further properties of the core and competitive equilibria. It used constructive proofs based on TTC.[5] Fundamental properties of TTC were first demonstrated in those proofs.

This paper is organized as follows. In Sections 1.1 and 1.2 Al will give an informal overview of TTC and of this paper, starting with his first encounter with Shapley and Scarf (1974), including some verbal proofs (in the spirit of the proof that TTC always produces an outcome in the core). Section 1.2 fast-forwards through some of TTC's developments as a tool of market design. Section 2 gives an overview of the adaptation of TTC to assignment with priorities, while Section 3 focuses on generalizing TTC to many-to-one assignments. In Section 4, Thayer will give an overview of how he and others teach TTC in classes on mechanism and market design (the uninterested reader can safely skip this section), and Section 5 concludes.

[4] Shapley and Scarf (1974) considered the core to be defined in the space of payoff vectors, rather than in the space of allocations of goods, as we have defined it. Scarf (2009) gives an illuminating description of the history of Shapley and Scarf (1974) and also the interaction with Gale that led to TTC (after which, he recalls, he wrote to Gale saying "I'm terribly sorry to have tell you that there is nothing wrong with your argument".).

[5] One can see further when standing on the shoulders of giants, particularly if one has been given a better telescope.

1.1. Overview (by Al)

I encountered Shapley and Scarf (1974) almost immediately when it appeared in the inaugural issue of JME, when it was brought to my attention as I was speaking about game theory during my job market fly-out at the University of Illinois in the Spring of 1974.[6] If memory serves, I read it on my flight home to California and thought about it before starting work at the U of I in the Fall. There, I began my collaboration with Andy Postlewaite, who had also just arrived as a new assistant professor.

I was struck by the fact that Shapley and Scarf had defined a *family* of TTC algorithms, since their definition was agnostic about the order in which cycles of agents and items were removed from the market. They showed that any order yields an allocation in the core. One of the things I noticed is that, if all agents have strict preferences over all items (i.e. if no agent is indifferent between any two items), then any ordering of the TTC algorithm yields the same allocation as any other ordering. This implies that there is a unique outcome in the "core defined by weak domination" of the game, which is how the result appeared in Theorem 2 of Roth and Postlewaite (1977).[7]

The direct proof that (in this simple model) all top trading cycles yield the same outcome when preferences are strict is straightforward. Here is a sketch of one approach (see Carroll, 2014 for another).

Theorem 1 (*Roth and Postlewaite 1977*)**.** *When all agents have strict preferences, TTC produces the same outcome regardless of the order in which top trading cycles are removed.*

Proof. Consider the directed graph whose nodes are agents together with their initial endowments, and whose directed edges connect every agent to their most preferred item. (That is, this is the graph we get when every agent points to their first choice item.) It has at least one cycle and possibly multiple cycles. Label with a "1" each node that occurs in one of these "top" cycles. In any ordering of TTC, the agents in a given cycle labeled in this way must all be removed at the same time, in the cycle of which they are a part, because it will remain a cycle no matter what cycles are removed earlier. So every agent labeled "1" receives the item they are pointing to in any ordering of cycle removal in TTC.

Nodes are labeled with a number k greater than 1 by having all agents who have not been labeled with a smaller number point to their first choice from the items that have not yet been labeled $1, \ldots, k-1$. Label with a "k" each node that is part of a cycle in the resulting graph. As above, in any ordering of top cycle removal, TTC assigns each of these agents labeled "k" the item labeled "k" that she is pointing to. Hence all agents and items can be labeled uniquely in a way that corresponds to which cycle they will be part of and which item they will be assigned, for any order in which top trading cycles are removed.[8] □

[6] My soon-to-be colleague Walter Rom pointed it out to me.

[7] The core is now generally defined as the set of undominated outcomes in a game, and so the definition of domination was important in the model of Shapley and Scarf (1974). Normally we say that an outcome x in this game is dominated if there is some coalition S whose members, by trading their initial endowments among themselves, can each get an object that they strictly prefer to the one they get at x. But we could also say an outcome x is "weakly dominated" if the members of S can (only) achieve an outcome that they like at least as well as what they get at x, as long as at least one member of S strictly prefers the new outcome to x. The importance of this distinction arises from the fact that there is no divisible commodity with which to "encourage" coalition members who are, in fact, indifferent between two outcomes, so the core defined by weak domination may be a strict subset of the core defined by strong domination. (See Yilmaz and Yilmaz, 2022 for some properties of TTC when the core defined by weak domination is empty, and see Feng and Klaus, 2022 for consideration of preference domains on which it is nonempty.).

[8] For independence of order for other mechanisms see Ergin (2000).

Shapley and Scarf (1974) noted that their paper was related to that of Gale and Shapley (1962) on the "marriage problem" because both papers dealt with the core of a market with indivisible items. Gale and Shapley's marriage problem was "two-sided:" it involved two distinct sets of agents, e.g. students and schools, who could each interact only with one another, while Shapley and Scarf's house market was "one-sided:" any agent could trade houses with any other. I was interested in both models because they each proposed an algorithm for constructing an outcome in the core (the deferred acceptance algorithm and TTC, respectively). Both of these algorithms took as input the preferences of the agents. As I began to think of these algorithms as engines that might drive a centralized clearinghouse for such marketplaces, I asked how the market maker could expect to learn those preferences. In Roth (1982b) I showed that, when TTC is used as a direct revelation mechanism, in which players' strategies are choices over what ordinal preferences to state over the items to be allocated, it is a dominant strategy for every participant in Shapley and Scarf's housing market to state their preferences truthfully. (The answer for the participants in Gale and Shapley's marriage market when the deferred acceptance algorithm is used is a bit more complex.[9])

Theorem 2 (*Roth 1982b*)**.** *It is a dominant strategy for every player to reveal their true preferences when allocations are made by TTC using stated preferences.*

Here's a new, simple proof that makes direct use of Theorem 1.

Proof (*Adapted From Gonczarowski et al. 2022*)**.** By Theorem 1, it will be sufficient to show that for each player i, there is *some* order of removing cycles in TTC such that he or she cannot get an assignment preferred to the one that would result from revealing true preferences. Fix a particular player i, and consider a version of TTC that first removes all cycles that do not contain player i. So player i's preferences will only be relevant when the graph is reached such that any cycle must contain player i. That is, TTC will assign to player i any one of the remaining items that he or she points to as his or her first choice among the remaining items. So player i can do no better than to point to his or her true first choice.[10] □

Gonczarowski et al. (2022) suggest that this argument makes transparent that it is a dominant strategy for players to state their true preferences in that it presents them with a menu (consisting of the items remaining when it is their turn to choose), and their own stated preference has no influence on the menu with which they will be presented. Their stated preference is used only to choose whichever item they claim is their first choice from that menu. This might therefore be a superior way to present the algorithm, to make clear that no advantage can be gained by mis-stating preferences.[11]

Note that the discussion so far combines what in the 1970's were still mostly thought of as cooperative game theory and non-cooperative game theory. That TTC produces an outcome in the core concerns what coalitions can and cannot do (no coalition can improve through its own actions what its members receive from TTC, since it is an outcome in the core of the game), although the model abstracts away from what specific actions individuals and coalitions might take. Theorem 2 is about strategies, and in this case, when we consider TTC as a direct revelation mechanism in which players' strategies are to state ordinal preferences, it is a dominant strategy to reveal preferences truthfully.

This use of both coalitional and strategic models turns out to be important in practical market design, since what is often being designed is a marketplace that will operate as a mechanism in a larger economic environment. Marketplace designers can specify the strategies available in the marketplace, but in the larger environment, in which agents have large and largely unobservable strategy sets, designers can only consider incentives and not detailed strategies (see Roth and Wilson, 2019). This is why it can be desirable to specify a mechanism whose strategic equilibrium in the game induced by the mechanism is contained in the core of the game (since if outcomes are outside the core of the larger game, the coalitions that are tempted to circumvent the mechanism might find strategies that allowed them to do so.)

1.2. The turn towards applications

Shapley and Scarf (1974) had used "houses" as a metaphor for the indivisible goods with which agents in their model are initially endowed. As my students never tired of pointing out to me, money solves a lot of the problems that arise in a model in which houses can only be traded for one another, which is why actual houses are bought and sold for money (which is divisible), and not bartered through TTC.[12] But in 1982 I moved to the University of Pittsburgh which was one of the biggest centers of organ transplantation in the world. When I taught the housing model, I often noted that we could interpret the agents and houses as being patient–donor pairs, with the donor interested in donating a kidney to the patient and both of them interested in finding the best available kidney for that patient.[13] I was not then yet thinking of kidney exchange as a practical application, and indeed actual kidney exchanges were still far in the future.

Abdulkadiroğlu and Sönmez (1999) were thinking explicitly of a market design application of a different sort in which money does not customarily pass between participants, namely the assignment and reassignment of college dormitory rooms to students. In their model, some rooms are unoccupied: they have just been vacated by graduating seniors. Some students do not yet have a room: they are entering freshmen, newly arriving on campus. And some students, already enrolled but not yet ready to graduate, have a room, but might be interested in moving to a room that they prefer. Abdulkadiroğlu and Sönmez pointed out that TTC could be adapted to this model by choosing an arbitrary priority order for the students and then constructing an initial directed graph of rooms and students in which each student pointed to their favorite room, each occupied room pointed to its resident student, and each unoccupied room pointed to the student with the highest priority. Then there would be at least one cycle (and every cycle containing an unoccupied room would contain the highest priority student). After such a first top trading cycle was removed and its rooms allocated accordingly, the process could be repeated with the remaining students and rooms. Further, this implementation of TTC would produce an allocation in the core and make it a dominant strategy for students to state their true preferences.

[9] Roth (1982a) demonstrated that there is no core-selecting mechanism that always makes it a dominant strategy for both sides of the marriage problem to state their true preferences, but that it is possible to make it a dominant strategy for one side to do so (namely the side that proposes in the deferred acceptance algorithm). Dubins and Freedman (1981) also proved the second of those results (extended to coalition strategyproofness) and Bergstrom and Manning (1983) proved the first. See Roth (2008) for some history of the deferred acceptance algorithm and its uses.

[10] See also Svensson (1999), and see Bird (1984) on strategyproofness for coalitions.

[11] TTC is not "obviously" strategy proof in the sense of Li (2017), and Troyan (2019) points out that this is the case for most preference profiles. When preferences are single-peaked in one dimension (e.g. size of houses), Bade (2019) suggests a simple presentation.

[12] For house allocation without money but in which houses are divisible see Athanassoglou and Sethuraman (2011).

[13] The passage of the 1984 National Organ Transplant Act, which forbids the purchase from donors of organs for transplant, made the housing model in which money was unavailable particularly appropriate for kidneys. Kidney transplants became more familiar to many after Dr Joseph Murray, who performed the first successful human kidney transplantation in 1954, received the 1990 Nobel Prize in physiology or medicine.

Abdulkadiroğlu and Sönmez also introduced an alternative implementation of TTC: "you request my house - I get your turn (YRMH-IGYT). It proceeds as a serial dictatorship[14] until the dictator chooses a house that is already owned. At this point, the house's owner becomes the dictator (and the previous dictator is next in line). This process is repeated if the new dictator selects a house with an owner. The process can even loop, and loops turn out to be trading cycles. The authors show that this produces the same outcome as TTC, and is strategyproof, Pareto efficient, and individually rational.[15]

The model of dormitory rooms seemed to me to offer an analogy to organ transplantation, since deceased donor organs can be viewed as roughly analogous to unoccupied rooms; patients with living donors who wish to donate a kidney to them are similarly analogous to students who already occupy a room; and patients without an intended donor are like students who do not yet have a room.

But this did not become a practical concern for me until I moved to New England, where in 2000 the first kidney exchange in the U.S. between two incompatible patient–donor pairs was performed at the Rhode Island hospital. After that, we eventually wrote Roth et al. (2004) which proposed a TTC-based model of kidney exchange involving both cycles and chains. Although never adopted in practice, that paper has provided a conceptual model for many subsequent practical developments.[16]

One reason that TTC itself was never applied to kidney exchange is that it could produce long cycles that would exceed the logistical abilities of transplant centers to coordinate with one another, particularly when all the surgeries involved had to be performed simultaneously.[17] So the New England Program for Kidney Exchange initially conducted only pairwise exchanges (see Roth et al. 2005b,a), and gradually longer cycles, and short simultaneous chains beginning with a nondirected donor were introduced. Somewhat later the Alliance for Paired Kidney Donation (APKD) created the first longer, non-simultaneous chains beginning with a non-directed donor.[18] Today, kidney exchange is organized in terms of cycles and relatively long non-directed donor-initiated chains, although the matches are largely computed with the help of integer programs (see Ashlagi and Roth, 2021).[19]

Another application of TTC is to school choice in some cities. An important distinction in this connection is whether the assignment of students to schools is an interaction between families and schools in which both sides have preferences (a two-sided market), or whether the schools are passive, and the problem is simply to allocate school places to students based on student/family preferences.

In New York City both schools and families had preferences, and our team recommended and the city adopted a mechanism based on the deferred acceptance algorithm for two-sided matching, which they used for the first time in 2003 (Abdulkadiroğlu et al., 2005, 2009). Nevertheless, for several years, NYC high schools used a version of TTC to reassign students following successful appeals of their assignments, in which case TTC allowed them to trade places with one another in some circumstances, reassigning students to suitable schools without increasing the enrollment in those schools.[20]

But in Boston, the schools were passive: only the families had active choices to make. Boston Public Schools assigned a priority at each school to each child in a fixed way, while parents were invited to submit preferences over schools. A highly influential paper by Abdulkadiroğlu and Sönmez (2003) studied the prevailing school choice system in Boston (which used a manipulable "immediate acceptance" algorithm), and discussed how both the deferred acceptance algorithm and an adaptation of TTC could be an improvement.[21] The adaptation of TTC can be described as follows. Initially, all students and schools are available. At each step, each available student points to their top choice among all available schools. If a student has no acceptable school among the remaining ones, she is assigned to herself and becomes unavailable.[22] Each available school points to the student who has the highest priority for the school among all available students. When a top trading cycle is formed, every student in the cycle is assigned to the school they are pointing to, and the remaining number of positions in each school is decreased by one each time a student is assigned to it.

We recommended both deferred acceptance and TTC as possibilities (see Abdulkadiroğlu et al., 2005, and in 2005 BPS voted to adopt the deferred acceptance algorithm (Abdulkadiroğlu et al., 2006). However, TTC was adopted in 2012 as the primary school choice algorithm for the New Orleans Recovery School District, where (as in Boston) schools are not strategic participants in school choice.[23]

[14] A serial dictatorship mechanism specifies an order over agents, and then lets the first agent receive her favorite object, the next agent receive her favorite object among those that remain, etc.

[15] Sönmez and Ünver (2010) provides a further characterization for the YRMH-IGYT mechanism, while Ekici (2013) characterizes the interim allocations of this process as being "reclaim proof".

[16] Indeed, some of the most recent theoretical papers on kidney exchange still use TTC as a benchmark. See e.g. Klimentova et al. (2023) and Nicolò et al. (2023). Biró et al. (2023) considers the housing exchange model of Shapley and Scarf (1974) and the core (defined by weak domination) results of Roth and Postlewaite (1977) and shows that the core *respects improvements*: if an agent's house becomes more desirable to some of the other agent's, then that agent's outcome weakly improves. The authors then apply this idea to kidney exchange, where respecting improvements is an important property. Schlotter et al. (2021) is another recent paper that considers respecting improvements in the house exchange model. (Hatfield et al., 2016 observes that TTC does not respect improvements in colleges in the many-to-one college admissions problem.).

[17] Before kidney exchange, transplant centers almost never needed to coordinate with one another, but kidney exchange involved a big change in the organization of transplants involving patients at different centers (Roth, 2024).

[18] A non-directed donor is someone who is willing and able to donate a kidney but does not have a particular recipient in mind. Such a donor can donate to the patient in some incompatible patient–donor pair, whose donor can pass it forward, etc. Because each patient–donor pair receives a kidney before they give one, there is not a risk that some pair will donate a kidney and then not receive a transplant. This reduced risk allows these chains to be non-simultaneous. See Roth et al. (2006), Rees et al. (2009), Ashlagi et al. (2011b,a).

[19] In the Indian state of Gujarat, where non-directed donor initiated chains are forbidden by regulation, the pioneering transplant nephrologist Dr. Vivek Kute has managed to arrange long, non-simultaneous, non-anonymous kidney exchange cycles of up to ten patient donor pairs, so under the right circumstances long cycles may be feasible, see Kute et al. (2021).

[20] Atila Abdulkadiroğlu and Parag Pathak were instrumental in the implementation of the NYC assignment. They have established themselves as giant contributors to the science of school choice, and not just to its practice, contributing new methods to the study of school choice outcomes. For example see Abdulkadiroğlu et al. (2016, 2017).

[21] Chen and Sönmez (2006) reports a related experimental study.

[22] Once the algorithm concludes, such a student is assigned administratively to a school with a vacant seat.

[23] The design team in New Orleans was the original NYC design team of Abudlkadiroglu, Pathak and Roth, together with Neil Dorosin, who had been the Director of High School Operations at the New York City Department of Education when we worked with NYC. Later, we supported Neil in creating the nonprofit Institute for Innovation in Public School Choice (IIPSC), which helped innovate school choice in a number of American cities. Our experience in IIPSC taught us that when we worked with school districts, we should help design not just a mechanism, but also the communication package that explained that mechanism, and in New Orleans, the adoption of TTC was preceded by an article in the local newspaper explaining it. Perhaps if we had already known of the proof of Theorem 2 we could have explained it more clearly. Regarding New Orleans, see also Abdulkadiroglu et al. (2020). And see Ortega and Klein (2023), which compares TTC to rank efficiency in Budapest data.

2. Assignment with priorities

In Shapley and Scarf's model, the traders own the objects being traded, and so TTC is well defined in terms of who can trade what to determine a final allocation of goods. In school choice, students do not initially own places in schools, but the assignment rules typically give students different priorities for different schools, and we will see how high priority can be interpreted as a kind of temporary ownership of a place in a school.

There are closely related problems, sometimes called simply "assignment problems", in which the goal is to assign objects to agents who do not start off with any kind of ownership of the objects.[24] When the value of each object to each agent is known, these are often simply optimization problems, whose goal may be to maximize the total value created.[25] But when the value of each object to each agent is the private information of each agent, assignment problems become mechanism design problems that must elicit the necessary information about values, as part of the assignment process.[26]

A venerable mechanism for assigning objects to agents, in this case, is a serial dictatorship (SD): the agents are ordered in some way, after which the agents, in order, choose their most preferred object from those that have not yet been chosen by earlier agents. This method is strategyproof: no agent can do better than to choose his most preferred object. It is also Pareto efficient.[27]

Note that SD can be interpreted as TTC on the graph in which each agent points to his most preferred object and each object points to the agent with the currently highest priority. (Then any object pointed to by that agent forms a top trading cycle consisting of that agent and object, and after that agent and object leave the market, all objects point to the next highest priority agent who can choose among the remaining objects, and so on.)

It turns out that TTC and SD are closely related even when agents are not ordered in priority. For a given set of objects to be allocated, consider the lottery over outcomes that results from the random assignment of the initial endowments to the players followed by TTC, and compare it to the lottery over outcomes that results from a random ordering of the agents followed by serial dictatorship. A striking result in the literature, proved independently by Knuth (1996) and Abdulkadiroğlu and Sönmez (1998), is that these two different lotteries result in the same *ex-ante* distribution over outcomes for the agents.

Theorem 3 (*Knuth 1996 and Abdulkadiroğlu and Sönmez 1998*)**.** *For any housing allocation problem, for any Pareto efficient outcome μ, the number of endowments where the outcome of TTC is μ is the same as the number of orderings where the outcome of a serial dictatorship is μ.*

We prove this below using the earlier approach of Knuth (1996), which is less well-known in the matching literature. In the proof, we assume that preferences are strict; however, the same result holds when preferences can be weak so long as ties are broken in a stochastically consistent manner (see Long and Tian, 2014).[28] Knuth starts with a permutation of the agents (the order of dictators), and constructs a mapping to trading cycles. Once the trading cycles are determined, he sets endowments so that these are the top trading cycles that occur. The proof is constructive and is best understood through an example. To help illustrate, we will work through the following example while we formally prove the result.

Example 1. In our running example, there are nine agents with the permutation π and relevant preferences given below.[29] Our goal is to map each permutation π to an endowment ω such that $TTC(\omega) = SD(\pi)$. We establish the result by showing that this mapping is one-to-one. The boxes indicate the allocation under $SD(\pi)$.

$\pi = (5, 3, 4, 9, 1, 8, 2, 7, 6)$

Preferences								
P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9
$\boxed{h_3}$	$\boxed{h_4}$	$\boxed{h_1}$	h_1	$\boxed{h_9}$	$\boxed{h_6}$	h_5	h_9	$\boxed{h_2}$
			$\boxed{h_5}$			h_3	$\boxed{h_7}$	
						$\boxed{h_8}$		

Proof. Fix strict preferences P.

The key to Knuth's proof is a systematic way of labeling and arranging cycles. Roughly speaking, Knuth writes a cycle with the highest numbered agent first (called the cycle-leader) and arranges cycles in increasing order by cycle leader. More generally, given a list of integers, $i_1, i_2, \ldots, i_k$; a set of integers I such that $i_1 \in I$; and an assignment μ, we define a cycle partition process. Knuth requires that a cycle leader be in the set I, the reason for which will be explained later. Let n be the first index such that $i_n \in I$ and $i_n > i_1$ (setting $n = k + 1$ if not such index exists). Then $(i_1, \ldots, i_{n-1})$ is our first cycle, and we define the initial endowment of each i_l to be the assignment of its predecessor: formally, $\omega(i_l) = \mu(i_{l-1})$ where it is understood that $\omega(i_1) = \mu(i_{n-1})$. We remove $\{i_1, \ldots, i_{n-1}\}$ and repeat. For illustration, the integers $\boxed{4}, 2, \boxed{6}, \boxed{3}, 8, \boxed{9}$ (where the boxes indicate the set I) is partitioned into cycles $(4, 2)$, $(6, 3, 8)$, and (9).

Similarly, we define an inverse process for transforming a partition of cycles into a list of integers. Given a cycle and a set of integers I containing at least one integer of the cycle, we define the cycle leader to be the largest integer of the cycle contained in I. Given a set I and a partition of cycles, our integer transformation process positions the cycles in increasing order by cycle leader and writes each cycle with the cycle leader first. For example, the cycles $(\boxed{3}, 8, \boxed{6})$, $(\boxed{9})$, and $(2, \boxed{4})$, with set I indicated by boxes, would be written $(\boxed{6}, \boxed{3}, 8)$, $(\boxed{9})$, and $(\boxed{4}, 2)$. They would then be ordered $(\boxed{4}, 2)$, $(\boxed{6}, \boxed{3}, 8)$, and $(\boxed{9})$, resulting in the list of integers $4, 2, 6, 3, 8, 9$.

Fix a permutation π. In our running example, the permutation is $\pi = (5, 3, 4, 9, 1, 8, 2, 7, 6)$. Let $\mu = SD(\pi)$, i.e. the allocation resulting from running a serial dictatorship with agents choosing in the order π. Initialize X to be all agents, and H to be the set of all houses. Define Y^1 to be the set of agents i such that $\mu(i)$ is her favorite house. Initialize

[24] Shapley and Scarf's housing model is often spoken of as a "one-sided" model because (in contrast to Gale and Shapley's marriage model) any agent can transact with any other agent. Assignment models in which agents are to be assigned objects that no one initially owns are also referred to as one-sided, in a slightly different sense, because only agents (and not objects) have preferences and welfare.

[25] See Kuhn (1955) on the assignment problem (and for a 50th anniversary celebration of that paper see Pentico (2007)). See Bogomolnaia and Moulin (2004) about allocating probability shares for ex-ante efficiency in expected utility. For a two-sided matching problem related to the marriage problem with cardinal preferences, see Shapley and Shubik (1972), and Lock et al. (2024) which studies competitive equilibrium in a housing model with money.

[26] See Hylland and Zeckhauser (1979).

[27] See, for example, Roth and Sotomayor (1990) for a discussion of serial dictatorships.

[28] We thank Atila Abdulkadiroğlu and Aram Grigoryan for this citation. Consistency for Long and Tian (2014) includes a requirement that tie breaking (to turn weak preferences into strict preferences) should be the same for both the TTC agents and the serial dictators, and independent of the order of serial dictatorship. There remain open questions about how the theorem might generalize to weak preferences and decentralized TTC (e.g. in which tiebreaking could depend on the history of prior play, etc., and in which a dictator's tiebreaking could depend on the set of items from which he chooses.

[29] For notational convenience, we suppress the full preferences. We define an endowment of houses ω to be a permutation of houses $\{1, \ldots, n\}$. Similarly, an *ordering* of agents is a permutation π of agents $\{1, \ldots, n\}$. The outcome $TTC(\omega)$ is the assignment made by TTC with endowments ω. The outcome $SD(\pi)$ is the assignment made by a serial dictatorship with ordering π.

$A := Y^1$. In our example, $A = \{1, 2, 3, 5, 6, 9\}$. We will refer to A as the anchors as they pin a trading cycle to a particular round. Note that a necessary, but not sufficient, requirement for an agent to be in a top trading cycle is that they receive their first choice. In our example, the first two dictators, 5 and 3, receive their top choice, but the third, 4, does not. Agent 4 cannot be part of a top trading cycle, so it is natural to make 5 and 3 our top trading cycle. Formally, let k be minimal such that $\pi(k+1) \notin Y^1$ (letting $k = n$ if all agents are in Y). Partition $\pi(1), \dots, \pi(k)$ into cycles using the partition process with set A and assignment μ. In our example, the agents 5,3 will be partitioned into the trading cycle $(5, 3)$ by endowing 5 and 3 with h_1 and h_9, respectively. We remove the agents and their assignments.

We must be careful with the remaining agents who receive their top choice. Such an agent might form a trading cycle too soon for our ordering. We will call these "balloons" as they need an "anchor" so that they do not fly away. In this example, our balloons are $\{1, 2, 6, 9\}$.

For round $k + 1$, we define Y^{k+1} to be the set of agents i such that $\mu(i)$ is her favorite remaining object, and we define $A = Y^{k+1} \setminus Y^k$. Note that the first agent is in A as the current dictator is always assigned her favorite remaining object, but by construction, her assignment was not her favorite remaining object in the previous round. Knuth requires that a cycle leader be an anchor in order to ensure that the trading cycle occurs in the right round. We now repeat until no agents remain.

In our example, after removing agents 3 and 5 and their objects, agents $\{1, 2, 4, 6, 8, 9\}$ all receive their favorite remaining object. The anchors, those who preferred a different object in the previous round, are $A = \{4, 8\}$. We identify the first dictator who does not receive her top choice and proceed as before. The only difference is we require a cycle leader to be an anchor. Otherwise, it is the same: we start a new cycle when we reach a higher-numbered cycle leader. The first agent who does not receive her favorite object is 7. Therefore, the agents to be partitioned are $(4, 9, 1, 8, 2)$. The agents 1, 9, and 2 are balloons and must be anchored. The anchors are 4 and 8. Therefore, $(4, 9, 1)$ is one cycle, and $(8, 2)$ is a separate cycle as 8 is a higher-numbered anchor. We make these trading cycle by endowing 4, 9, and 1 with h_3, h_5, and h_2, respectively. Similarly, the endowments of 8 and 2 are h_4 and h_7, respectively. The final cycle, $(7, 6)$ is formed analogously. Agent 7 is endowed with h_6, and 6 is endowed with h_8.

We summarize the endowments below:

$$\begin{pmatrix} \omega(1) & \omega(2) & \omega(3) & \omega(4) & \omega(5) & \omega(6) & \omega(7) & \omega(8) & \omega(9) \\ h_2 & h_7 & h_9 & h_3 & h_1 & h_8 & h_6 & h_4 & h_5 \end{pmatrix}$$

The inverse mapping is analogous. Given endowments ω, let $C_1, \dots, C_k$ be the top trading cycles in the current round of TTC. For any round, we define I to be the agents in these cycles who are pointing to their house for the first time. Note that every cycle contains at least one member in I or else it would be a cycle in the previous round. Use the integer transformation process defined above; append the result to our previously constructed list of integers; and then remove the agents and houses and repeat.

In our example, given the endowments above, there is one top trading cycle: $(3, 5)$. The cycle can be written $(3, 5)$ or $(5, 3)$ but our convention is to have the highest ranked anchor[30] be the cycle leader. Therefore, we write the cycle as $(5, 3)$, and we remove these agents and these objects. In the second round of TTC, there are two cycles: $(1, 4, 9)$ and $(2, 8)$. An agent is an anchor if her favorite object in this round was not her favorite in the previous round. In this case, 4 and 8 are anchors. Therefore, we rewrite the cycles as $(4, 9, 1)$ and $(8, 2)$, and we arrange the cycles so that they are increasing in the cycle leader. In this case, $(4, 9, 1)(8, 2)$. In the final round, there is one cycle, $(6, 7)$, and the anchor is 7, so we write the cycle as $(7, 6)$. The cycles, listed in order, are:

$$(5, 3)(4, 9, 1)(8, 2)(7, 6)$$

and we can see that we have recovered the permutation we started with. This is true in general. It is straightforward to verify that the two processes invert each other. Moreover, the same assignment is made by both $SD(\pi)$ and $TTC(\omega)$. Therefore, there is a one-to-one mapping between permutations and endowments that yield the same allocation, establishing the result. □

This kind of result had a practical application in NYC school choice, in which students are assigned to schools in a main round followed by a supplemental round. The main round is viewed as a two-sided matching problem and uses a mechanism based on the deferred acceptance algorithm (Abdulkadiroğlu et al., 2005 and Abdulkadiroğlu et al., 2009). The supplemental round is comparatively simple: unassigned applicants are assigned to available seats based on strict preferences submitted by the students, with schools indifferent between students in this round.[31] But lotteries have to be conducted to determine which students can be admitted when the number of students who might be matched to a school exceeds its available capacity. It turns out that the choice of whether to use a random serial dictatorship or TTC from random endowments is equivalent to the choice of conducting one lottery for all schools or having each school conduct its own lottery. Anecdotally, a different lottery at each school seemed fairer to many participants, including some policy makers at the NYC Department of Education. Pathak and Sethuraman (2011) generalize the results of Knuth (1996) and Abdulkadiroğlu and Sönmez (1998) and show that in the supplemental round, the distribution over assignments is the same whether schools use a single lottery for all schools or multiple lotteries, one for each school.[32]

2.1. Priorities and property rights

In Shapley and Scarf's model, the traders own the objects being traded, and can transfer ownership by exchanging their initial endowment and receiving a different object. One way to think of the way priority works in serial dictatorship is that the player with the highest priority (i.e. the current dictator, whose turn it is to choose among all the objects) "owns" all the objects, until he chooses one and exits the market with it. At that point, ownership of all the objects passes to the player with the next highest priority, i.e. to the next dictator. Similarly, in school choice (when school places are passive objects), a student with the highest priority at a school or schools with at least one vacant position can be viewed as "owning" those positions until he or she has been assigned to one (at which point "ownership" of remaining positions at those schools shifts to whichever other students then have the highest priority for them).

Pápai (2000) significantly generalized TTC by considering more broadly how property rights of this sort could be transferred from one agent to another.[33] Papai introduced a class of mechanisms she called *hierarchical exchanges*.

A hierarchical exchange generalizes the housing exchange model in that a trader can own more than one object. A hierarchical exchange rule is defined by how the objects not used in a top trading cycle, but owned by one of its traders, are "inherited" by the remaining

[30] This is the first round, so both agents are anchors.

[31] The supplemental round in NYC is of course a many-to-one matching model as more than one student can be assigned to the same school.

[32] The equivalence in the distribution of outcomes is a remarkably general result. In addition to Pathak and Sethuraman (2011), Lee and Sethuraman (2011), Carroll (2014), and Bade (2020) are all papers that show that this equivalence extends to more general models.

[33] Other generalizations allow preferences to be expressed over certain attributes of objects rather than over individual objects. Two strategyproof, Pareto efficient, and weak core mechanisms were introduced independently by Alcalde-Unzu and Molis (2011) and Jaramillo and Manjunath (2012). These algorithms are generalized first by Aziz and De Keijzer (2012) and then further generalized by Saban and Sethuraman (2013) and Plaxton (2013).

agents. Priority in school assignment is an example of this. However, hierarchical exchanges are defined for far more general inheritance rules. In particular, the inheritance can depend on the students that were assigned earlier and the schools they were assigned to. However, once a student inherits a school, she owns the school until she is part of a top trading cycle. Papai's model generalizes the housing exchange model, house allocation with existing tenants, school assignment,[34] and other models.

In the spirit of the core, Papai considers ways in which a group of traders might attempt to circumvent an apparently strategyproof mechanism. One way would be to misreport their preferences in a coordinated manner. A mechanism f is *group-strategyproof* if for all R, there does not exist $M \subseteq N$ and $\tilde{R}_M$ such that for all $i \in M$, $f_i(\tilde{R}_M, R_{-M})\ R_i\ f_i(R)$ and for some $j \in M$, $f_j(\tilde{R}_M, R_{-M})\ P_j\ f_j(R)$. Roth (1982a) showed that TTC is strategyproof, and Bird (1984) extended this to group-strategyproofness.

A second way in which a group of traders might circumvent a mechanism is by reallocating their assignments among the group. An assignment mechanism f is *manipulable through reallocation* if two traders can misreport their preferences and then subsequently Pareto improve their allocation through a pairwise trade.[35] That is, if it is manipulable in this way there exists R, $i, j \in N$, and $\tilde{R}_i$, $\tilde{R}_j$ such that $f_j(\tilde{R}_i, \tilde{R}_j, R_{-i,j})\ R_i\ f_i(R)$; $f_i(\tilde{R}_i, \tilde{R}_j, R_{-i,j})\ P_j\ f_j(R)$; and $f_h(R) = f_h(\tilde{R}_h, R_{-h}) \neq f_h(\tilde{R}_i, \tilde{R}_j, R_{-i,j})$ for $h = i, j$. An assignment rule is *reallocation proof* if it is not manipulable through reallocation. Pápai (2000) demonstrates that hierarchical exchange is the unique class of mechanisms that satisfy core-like properties that give stability to the resulting outcomes.

Theorem 4 (*Papai 2000*)**.** *Suppose each agent has strict preferences over a set of objects. An assignment mechanism is group-strategyproof, Pareto-optimal, and reallocation-proof if and only if it is a hierarchical exchange rule.*

As part of her characterization of group-strategyproof mechanisms, Papai provides an elegant characterization of group-strategyproofness itself. Pápai (2000) connects group-strategyproofness to the property of nonbossiness, and provides a simple proof that TTC is group-strategyproof. A mechanism f is *nonbossy* if for all R, $i \in N$, and $\tilde{R}_i$, $f_i(\tilde{R}_i, R_{-i}) = f_i(R)$ implies $f(\tilde{R}_i, R_{-i}) = f(R)$. That is, a mechanism is nonbossy if a manipulation by player i that does not change player i's assignment does not change anyone's assignment.

Theorem 5 (*Pápai 2000*)**.** *When preferences are strict, a mechanism is group-strategyproof if and only if it is strategyproof and nonbossy.*

Proof. It is clear that group-strategyproofness implies strategyproofness and nonbossiness. To prove the converse, suppose for contradiction that mechanism f is strategyproof and nonbossy, yet there exists a group M and report $\tilde{R}_M$ such that for all $i \in M$, $f_i(\tilde{R}_M, R_{-M})\ R_i\ f_i(R)$ and for some $j \in M$, $f_j(\tilde{R}_M, R_{-M})\ P_j\ f_j(R)$. For every $j \notin M$, set $\tilde{R}_j = R_j$. Choose student $i \in M$ such that $f_i(\tilde{R}_M, R_{-M})\ P_i\ f_i(R)$, and define R'_i to rank $f_i(\tilde{R}_M, R_{-M})$ first, $f_i(R)$ second, and the other schools in an arbitrary order. Strategyproofness implies that $f_i(R'_i, R_{-i}) = f_i(R)$; therefore, by nonbossiness, $f(R'_i, R_{-i}) = f(R)$. Again, by strategyproofness, $f_i(R'_i, \tilde{R}_{-i}) = f_i(\tilde{R})$; therefore, by nonbossiness, $f(R'_i, \tilde{R}_{-i}) = f(\tilde{R})$. However, this is a contradiction as by assumption $f_i(\tilde{R}) \neq f_i(R)$. □

Pycia and Ünver (2017) finishes the characterization of group-strategyproof and Pareto-efficient mechanisms by enlarging the class of hierarchical exchange rules to allow for a new type of trader which they call a broker. They define i to *own* house h if, whenever all traders rank h first, i is the one assigned to h. Their innovation is to introduce a second type of control over an object. Trader i *brokers* object h if, whenever all traders rank h first, i receives her second choice. Intuitively, the broker can trade the object, but she cannot be assigned to it unless the brokered object is the only object remaining. Pycia and Ünver (2017) define a class of mechanisms called *trading cycles*. We refer a reader interested in a precise definition of trading cycles to the paper as it is technical and involved. However, intuitively, it proceeds in the same way as TTC. Each house points to its owner or broker. Each trader points to her favorite house for which she is not the broker. A cycle must exist, and each trader receives the house she is pointing to in the cycle. In a remarkable result, Pycia and Ünver (2017) show that this is the unique class of mechanisms that are group-strategyproof and Pareto efficient.[36]

Theorem 6 (*Pycia-Unver 2017*)**.** *Suppose each agent has strict preferences over a set of objects. An assignment mechanism is group-strategyproof and Pareto-optimal if and only if it is a trading cycles rule.*

3. Many to one matching

There are simple and elegant characterizations of TTC for the housing exchange model.[37] There are complicated but beautiful characterizations of TTC and its generalizations for the assignment model (Pápai 2000 and Pycia and Ünver 2017). However, the main application of TTC has been to school assignment. Abdulkadiroğlu and Sönmez (2003) introduce the canonical adaptation of TTC to the school assignment problem: each trader points to her favorite school and each school points to the student with the highest priority among the remaining students. While this is a natural adaptation of TTC to the school choice problem, a satisfying characterization of this algorithm has proved elusive.[38] Even though at first glance the assignment and school

[34] Abdulkadiroğlu and Sönmez (1999) is an earlier paper which made innovative use of priorities in adaptations of TTC. Balinski and Sönmez (1999) modeled centralized school assignment, and Abdulkadiroğlu and Sönmez (2003) explicitly adapts TTC to school assignment with priorities.

[35] Further, this misreport is safe as the trader is not harmed if she misreports her preferences, and her partner reports her true preferences.

[36] Yu and Zhang (2022) considers a housing exchange model in which restrictions on who can be assigned an object are part of the rules of the game (as opposed to a property of a mechanism).

[37] Ma 1994 shows that in the housing exchange model (when traders have strict preferences) that a mechanism is individually rational, Pareto efficient and strategyproof if and only if it is TTC. Sethuraman (2016) provides a simple and elegant proof of this result. See also Anno (2015) for a short proof of Ma's characterization. A weaker condition than Pareto efficiency is *pair-efficiency*: an outcome of a mechanism is pair-efficient if no pair of agents are better off trading their assignments. Ekici (forthcoming) shows that TTC is the unique individually rational, strategyproof, and pair-efficient mechanism. As pair-efficiency is a significantly weaker condition than Pareto efficiency, this characterization is correspondingly stronger than Ma's result. Ekici and Sethuraman (2023) provides a short proof of this characterization. Unanimity is also a weak efficiency condition: a mechanism satisfies *unanimity* if all agents are assigned to their favorite object whenever this is possible. Takamiya (2001) shows that TTC is the unique mechanism satisfying unanimity, individual rationality, and group-strategyproofness.

[38] When the school can only be assigned to one student, Abdulkadiroglu et al. (2020) show that TTC minimizes justified envy among strategyproof and Pareto-optimal mechanisms. Morrill (2013) characterizes TTC in terms of Pareto efficiency, strategyproofness, a basic fairness criterion, and a consistency condition. However, this characterization is somewhat technical and only holds when schools have unit capacities. Of course, schools do not have unit capacities. Dur and Paiment (2022) provided the first characterization of TTC with general capacities. Morrill (2014a) also provides a characterization under general capacities. More recently, Combe (2022) provides an alternative characterization of TTC in terms of minimal envy.(Combe et al., 2022 suggest a version of TTC as an alternative to current procedures in France governing the transfer of *teachers* from one school to another.).

assignment models look almost identical, objects with capacities greater than one dramatically expands the possibilities for Pareto efficient assignment. Here we discuss some of the variations of TTC for this environment.

The housing exchange model with strict preferences is mathematically simple because an agent can only be involved in one Pareto improving top trading cycle. The same is true for the assignment model with unit capacities. However, with capacities greater than one, there can be multiple Pareto improving cycles involving the same school and potentially overlapping sets of students. Put a different way, there is no longer a unique partition of students and objects into Pareto improving top trading cycles.

Hakimov and Kesten (2018) introduce **equitable top trading cycles** (hereafter ETTC) which is the first variation of TTC designed to reduce priority violations for a school assignment problem.[39] Under TTC, for a school s with capacity q_s, all q_s seats are allocated to the student with the highest priority, and only this student is allowed to trade. Under ETTC, each of the top q_s students are allocated a seat. The market now consists of student–school pairs (i, s) where i has one of the q_s highest priorities at s. Suppose student i's favorite remaining school is s'. Under TTC, i points to s'. Under ETTC, the student–school pair (i, s) points to (j, s') where j has the highest priority at s among all the student who have been allocated a copy of s'.[40] Hakimov and Kesten (2018) provide convincing experimental and simulation evidence that ETTC can reduce the number of priority violations relative to TTC.[41]

Morrill (2014b) introduces a second variation of TTC called clinch and trade. Clinch and trade is based on a simple observation. Consider a student whose favorite school is s and who has one of the q_s highest priorities at s. We can anticipate that under TTC this student will be assigned to s. Yet, if this student does not have the very highest priority at s, then she is allowed to trade her priority at a different school in order to be assigned to s. This trade is unnecessary to her assignment and potentially creates a distortion. Under clinch and trade, before school s points at a student, we assign all students whose favorite school is s and who have one of the q_s-highest priorities at s. Although a simple variation of TTC, this algorithm has interesting theoretical properties. Note that we can no longer describe the assignment process in terms of ownership as was done in Papai's hierarchical exchanges: an agent who clinches an object receives this object but never has the opportunity to trade the object.[42,43]

[39] This algorithm first appeared in Kesten (2004).

[40] This is the most intuitive implementation of ETTC, but as described in Hakimov and Kesten (2018), there are a variety of ways ETTC can be implemented. For example, if a school board wanted to restrict the ability of students to trade sibling or walk-zone priorities, this can easily be implemented in ETTC.

[41] Recently, Grigoryan (2023) introduced a simple modification of TTC that matches schools to higher priority students.

[42] There have also been generalizations of TTC designed to find an allocation in the multi-type housing market. In the multiple-type housing market (Moulin, 1995) an agent's endowment consists of more than one object (say, a presentation paper and a presentation date). Despite the similarity to the housing model, in this environment, there is no strategyproof, Pareto efficient, and individually rational mechanism (Konishi et al., 2001). Recent work by Di Feng and Klijn (2022), Feng (2023) introduced two natural variations of TTC tailored for the multi-type housing market. The papers characterize each algorithm in terms of natural strategy and efficiency properties.

[43] Another interesting variation of TTC is the Extended-Seat TTC introduced by Ueda et al. (2012). The Extended-Seat TTC applies when each object must be assigned to a minimum number of agents. This was applied to course assignment in Bichler et al. (2021).

4. A first lecture on TTC (by Thayer)

TTC is a standard component of any graduate or undergraduate course on market design.[44] In this section, I will go over the outline of the first lecture on TTC that I give in my field course. An uninterested reader can safely skip this section as the results have been covered earlier (although some of the proofs are different). Our intention for including it is twofold. Perhaps someone teaching a course on market design for the first time will find it useful. Second, while every researcher who has studied TTC knows that Roth (1982b) proves that TTC is strategyproof, even practitioners might not know that the foundational properties of TTC were established in Roth and Postlewaite (1977). This is because no theorem, corollary, or lemma in Roth and Postlewaite (1977) mentions TTC. Yet the proofs in Roth and Postlewaite (1977) are constructive, using TTC, and from these proofs, we learn the key properties of TTC.

I introduce the housing model and TTC exactly as described in the introduction, which is to say, exactly as described in Shapley and Scarf (1974). I always pause after saying "there must exist a cycle". It is enjoyable to watch the students' faces as they cycle through the same reaction I initially had: first, "really?!"; second, "oh yeah, of course."; and finally, "hunh, that's interesting". The existence of a cycle is simple yet has interesting implications. The following statement is true, but it did not occur to me until I saw TTC. In every Pareto efficient outcome, at least one person is getting her favorite house. Equivalently, if no one has her favorite house, then the outcome is not Pareto efficient. Even this simple statement feels like a paradox: it seems plausible to me that in a neighborhood, everyone might envy someone else's house!

My goal for the first class is to prove the following.

Theorem 7. *The outcome of Top Trading Cycles is:*

- *individually rational*
- *strategyproof*
- *Pareto efficient*
- *in the core (by weak domination)*
- *the unique outcome in the core.*

For my class, I assume that each trader has strict preferences over the houses. Note that Shapley and Scarf (1974) does not assume this. When agents may have indifference, the outcome of TTC is in the core defined by strict domination, but it may not be in the core defined by weak domination. In fact, the outcome may not even be Pareto efficient (Roth and Postlewaite, 1977). Of course, showing TTC is in the core implies that it is Pareto efficient and individually rational. However, I prefer to first show that TTC is individually rational and Pareto efficient in order to build up the students' intuition about TTC.

Proof.

TTC is Individually rational: Each step of TTC creates an interesting directed graph. Moreover, this graph is dynamic: it evolves as we progress through the rounds of TTC. But there is a structure to this movement. I ask the students to think of yourself as a participant in the problem and to recognize that your house is always pointing at you. It will never be a part of a cycle unless you are also part of a cycle. In other words, so long as you are present, your house is also available. Therefore, at all times, either you point to your own house or one you strictly prefer to it. Any cycle you form is either with your own house or one you strictly prefer: TTC is individually rational.

TTC is strategyproof: Understanding this graph also helps us understand why TTC is strategyproof. Your house never leaves before you do. Now suppose someone points to your house. This person will continue to point to your house until it is removed, which means she

[44] I am grateful to my advisor Larry Ausubel for many things, including introducing me to TTC during his field course on auction theory.

will continue to point to your house until you are removed. Therefore, although this directed graph changes from round to round, once there is a path from house h to agent i, that path remains until i is removed. Now, consider a variation of TTC biased towards an agent i. We fix every other agent's preferences, but in every round of TTC, we allow i to change her's if she desires. If i does not create a cycle, then who she points to is irrelevant. If she creates a cycle (by pointing to house h), then there must be a path from h to i. But as we mentioned, this path will remain so long as i remains. Therefore, there is no reason for i to "rush" and claim h before h is her favorite remaining object. She might as well wait and see if she can do better. Therefore, i always does best by pointing to her favorite remaining object: TTC is strategyproof.

The TTC outcome is Pareto efficient: I remind students of the simple argument that a serial dictatorship is Pareto efficient. This is clear to all students. The dictator cannot be improved. The second dictator either receives her favorite or else the dictator took it. The dictator does not want to switch. So, the second dictator cannot be improved, and so on. Then, it is clear that TTC follows the same logic. Agents in a top cycle receive their favorite house: they cannot be improved. Agents in a second round cycle either receive their favorite object, or else any object they prefer has been assigned in the first round, which we know cannot be part of a Pareto improvement, and so on. □

Next, I show my students that the TTC outcome is in the core. The proof I use is taken directly from the proof of Theorem 2 in Roth and Postlewaite (1977).

Theorem 8 (*TTC Undominated*)**.** *Suppose each trader has strict preferences over the houses. Then the outcome of TTC is in the core (defined by weak domination).*

Proof. Let x be the outcome of TTC with top trading cycles $S_1, S_2, \dots, S_p$.[45] Suppose for contradiction that x is blocked by some allocation y and some coalition C. Suppose $i \in S_k \cap C$ and $x_i = y_i$. We claim that $S_k \subset C$. If i owns x_i, then S_k is a trivial cycle and therefore the result is trivial. If $x_i = y_i$ but i does not own x_i, then the owner of x_i, call her j, must be included in C for the coalition to be effective. Similarly, the owner of x_j must be included in C. Repeating this argument, we find that all members of this Trading Cycle, S_k, must be included in C.

Choose an allocation y and a coalition C that is minimal in the following sense: there is no allocation y' and coalition C' that block x such that $|C'| < |C|$. Let S_l be the first trading cycle where there is an $i \in S_l \cap C$ such that $y_i \, P_i \, x_i$. Note that for any $k < l$, $S_k \cap C = \emptyset$. To see this, note that for all $j \in S_k \cap C$, $x_j = y_j$ as S_l was chosen to be the first cycle where a coalition member receives a different allocation. Therefore, by the above claim, if $S_k \cap C \neq \emptyset$, then $S_k \subseteq C$. However, since $S_k \subset C$ and $y_j = x_j$ for every $j \in S_k$, then $C \setminus S_k$ also blocks x (with outcome y). This would violate the minimality of C.

But now we have reached our contradiction. As $y_i \, P_i \, x_i$, by the definition of TTC, y_i must have been assigned in an earlier trading cycle. The owner of y_i must be included in C for the coalition to be effective, yet no member of an earlier cycle is included in C. This is a contradiction. □

A striking property of TTC is that any TTC allocation dominates any assignment. This is Lemma 1 in Roth and Postlewaite (1977), although it is presented in terms of competitive allocations.

Theorem 9 (*TTC blocks*)**.** *Let x be any outcome of TTC and y any alternative allocation. Then x weakly dominates y.*

Proof. Let x be the outcome of TTC with top trading cycles $S_1, S_2, \dots, S_p$. Let k be the first integer where there exists an $i \in S_k$ such that $x_i \neq y_i$. Then, by construction, the coalition $S_i \cup \dots \cup S_k$ is effective, and for every $j \in S_i \cup \dots \cup S_k$, $x_j \, R_j \, y_j$. Since $x_i \, P_i \, y_i$ (again by the way TTC is constructed), $S_i \cup \dots \cup S_k$ is a blocking coalition. □

This completes my final objective for this first class. Theorem 9 implies that the TTC outcome is the unique outcome in the core defined by weak domination as any other outcome is blocked. Interestingly, Theorem 9 also implies a very short proof that TTC is independent of the order in which cycles are chosen. I did not see a formal proof of this in the literature until Carroll (2014). Theorem 1 in the current paper is Al's "straightforward" proof of this fact. As the result has always been obvious to Al, perhaps this is why it did not appear in print. I sheepishly had to admit to Al that his proof was elegant and correct but, at least to me, not straightforward. Here is a far less elegant but simpler proof that TTC is independent of the order in which cycles are chosen.

Corollary 1. *The allocation made by TTC is independent of the order in which cycles are chosen.*

Proof. By Theorem 8, any outcome of TTC is undominated. By Theorem 9, any outcome of TTC dominates any alternative allocation. This is only possible if TTC generates a unique outcome. □

5. Conclusion

William Stanley (Jevons, 1876), in the first chapter of his book about money, famously noted that the use of money in exchange is a market design invention to overcome the big obstacle to barter, which is that (in the absence of money) trade between two parties can only occur if there is a "double coincidence" of wants. That is, the owner of a house who prefers another particular house can only exchange his house for that house if its owner also, coincidentally, wants exactly that trade. Kidney exchange between two patient–donor pairs depends on this coincidence: the donor's kidney in each pair must be compatible with the patient in the other pair. In this context, Roth et al. (2007) noted that the need for such a double coincidence can be somewhat relaxed if larger cycles of exchange are feasible and that a computerized clearinghouse can aid in finding efficient exchanges in a sufficiently thick market.

The success of kidney exchange, which sometimes now includes exchange in cycles involving many patient–donor pairs, suggests that computerized "smart market" clearinghouses can indeed facilitate barter on a reasonably large scale. Recall from Kute et al. (2021) that, in India, cyclic exchanges involving as many as ten pairs have been conducted. Note also that when the NYC Department of Education was conducting school choice appeals by TTC, cycles among as many as twenty students were sometimes identified and enabled. This allowed students who were assigned places in schools that turned out to be unsuitable for them to exchange places with other students so that each could be placed in an appropriate school without having to expand the capacity of those schools.

More generally, cyclic exchange, without the use of money, now seems feasible in ways that Jevons observed would have been exceedingly difficult to arrange before exchanges could be identified with computer-aided markets. It thus seems likely that TTC and its generalizations, which continue to enjoy active theoretical investigation fifty years after first being discussed, will motivate further novel applications in the years to come.[46]

[45] Note that if multiple cycles were chosen in a round, here we assign the same cycles, one after the other, in an arbitrary ordering.

[46] For example, Farajzadeh et al. (2023) consider refugee resettlement, specifically the placement of Ukrainian refugees in the United States following Russia's 2022 invasion of Ukraine. In counterfactual analysis, they find that using TTC would significantly improve the average rank of a refugee's match.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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